

# Interior volumes of extremal and $(1 + D)$ dimensional Schwarzschild black holes

Nilanjandev Bhaumik\* and Bibhas Ranjan Majhi†

Department of Physics, Indian Institute of Technology Guwahati,  
Guwahati 781039, Assam, India

July 14, 2016

## Abstract

It has already been shown for the Reissner-Nordström and Kerr black holes that the maximum interior volume enclosed by the event horizon ceases to zero in the extremal limit. We show here that if we start with an extremal black hole at the beginning, corresponding volume is non-zero. Interestingly, both for the extremal Reissner-Nordström and Kerr, this value comes out to be equal to one quarter of the horizon area, which is the entropy of the black hole. Next the same quantity is calculated for the  $(1 + D)$ -dimensional Schwarzschild case. Taking into account the mass change due to Hawking radiation, we show that the volume increases towards the end of the evaporation. This fact is not new as it has been observed earlier for four dimensional case. The interesting point we observe is that this increase rate decreases towards the higher value of space dimensions  $D$ ; i.e. it is a decelerated expansion of volume with the increase of spacial dimensions. This implies that for a sufficiently large  $D$ , the maximum interior volume does not change. The possible implications of these results are also discussed.

## 1 Introduction

Unlike the horizon surface area of a black hole, until some very recent developments, interior volume was not a subject of popular interest in black hole thermodynamics. The concept of interior volume in case of black holes contains some implicit difficulty in understanding. Since in curved geometry simultaneity surface loses its significance to be taken as the volume, we need to try to find an alternative definition of volume. Eventually multiple ways to define volume in curved geometry have been proposed [1]–[8]. Recently, Christodoulou and Rovelli suggested the volume to be the largest space like surface with spherical nature inside the black hole horizon [9]. From now we call this as CR volume. They showed that the maximum interior volume for a Schwarzschild case, is linearly proportional to advanced time. The same has been concluded for Kerr metric also [10]. Their formulation was completely classical. More recent analysis showed that this method carries vast significance while considering from thermodynamical point of view. It turned out that the CR formulation is useful to estimate phase space volume as well as the horizon entropy of a black hole [11]. Moreover, such a construction of volume, when Hawking radiation [12] is taken into account, has significant importance to resolve the information paradox problem (see [13]–[15], for details). It has been observed that the volume is always increasing with advanced time and therefore we shall have a large amount of volume at the end of the

---

\*E-mail: nilanjandevbhaumik@gmail.com

†E-mail: bibhas.majhi@iitg.ernet.in

evaporation [14]. Hence there is enough place to store the information. This has been further elaborated and clarified in [16]. Motivated by these facts and implications, here, in this paper we shall mainly concentrate on the definition of CR volume for some particular instances.

Until now all efforts to find the CR volume for an extremal black hole led to zero results (see [14] for Reissner-Nordström and [10] for Kerr black holes). This brings to a contradiction in the picture whereas extremal black holes are suggested to have non zero entropy according to the existing theory [17]–[19]. *How can be a nonzero entropy or information packet accommodated in a zero volume?* In this paper our first intention is to clarify this point. We shall show that if to calculate this CR volume, we take the extremal limit from the very first step and carefully choose the physically possible volume expression for constant  $r$  surface from the Eddington-Finkelstein line element we inevitably get a non zero CR volume. This will be done for both the extremal Reissner-Nordström (ERN) and extremal Kerr (EK) black holes. More surprisingly, both for these cases the rate at which volume changes with respect to advanced time ( $v$ ) comes out to be the same with its horizon entropy. Now for extremal cases, the horizon temperature is zero and hence there is no Hawking radiation. Therefore the macroscopic parameters of the metrics do not change with time and one can then integrate to obtain these volumes, which will be shown to be linearly proportional to advanced time.

Next we shall investigate another aspect. Our aim is to see how the CR volume behaves for higher dimensional black holes. Due to simplicity and availability of the relevant quantities, we shall restrict our analysis on a  $(1 + D)$ -dimensional Schwarzschild black hole. Taking into account the mass change due to Hawking radiation, it will be observed that the rate of change of maximum volume with advanced time for a particular value of spacial dimension is positive. Therefore, like the  $(1 + 3)$ -dimensional case [14], it is increasing. But the important observation, we shall note that, this rate is decreasing with the increase of dimensions of space. At sufficiently large value of  $D$ , the change is negligible and hence the maximum volume remains same. This will be shown by plotting the rate of change with  $D$ . Such fact will be further bolstered by calculating the total change of CR volume for the total evaporation time. We shall see that this is going to vanish in limit of  $D \rightarrow \infty$ .

The organization of the paper is as follows. Section 2 is divided into two subsections, calculating the CR volumes of the ERN and EK black holes, respectively. Next section contains the same for  $(1 + D)$ -dimensional black hole. Finally, we discuss our results and implications in section 4.

## 2 Extremal black holes

In this section, we shall evaluate the CR volume for two extremal black hole spacetimes: Reissner-Nordström and Kerr. It will be observed that a careful calculation leads to non-zero value. The implications of such situation will also be discussed.

### 2.1 Extremal Reissner-Nordström black hole

The metric of the  $(1 + 3)$ -dimensional Reissner-Nordström (RN) black hole is

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

The Hawking temperature is given by  $T = (r_+ - r_-)/4\pi r_+^2$ , where

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}. \quad (2)$$

Therefore, for the extremal case (i.e.  $T = 0$ ) we have  $Q = M$ . Hence the extremal Reissner-Nordström (ERN) black hole metric turns out to be

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3)$$

where

$$F(r) = \left(\frac{r-M}{r}\right)^2. \quad (4)$$

The horizon is at  $r_H = M$ .

Now to find the interior maximum volume one has to write the metric (3) in Eddington-Finkelstein coordinates:

$$ds^2 = -F(r)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (5)$$

where the advanced time is defined as  $v = t + r^*$  with  $dr^* = dr/F(r)$ . The maximum interior volume is the given by the metric on the  $r = \text{constant}$  surface <sup>1</sup>:

$$V = \int d\theta d\phi dv \, r^2 \sin\theta \sqrt{F(r)} = 4\pi \int^v r^2 \sqrt{F(r)} \, dv. \quad (6)$$

Note that here  $\sqrt{F(r)} = \pm(r-M)/r$  and since we are in the region  $r < M$  (i.e. within the horizon),  $V$  will be positive for the negative sign. Then it is given by

$$V = 4\pi \int^v r(M-r) \, dv. \quad (7)$$

Next to obtain the maximum one we need to maximize the integrand. It is easy to check that at  $r = M/2$ , the integrand is maximum. Then the maximum interior volume is calculated to be

$$V = \pi \int^v M^2 \, dv. \quad (8)$$

Therefore the rate of change of the mass with respect to the advanced time  $v$  is  $dV/dv = \pi M^2$ , which is one quarter of the horizon area. Moreover, note that in this case, like the non-extremal one [14], this quantity is positive and hence the volume is increasing with the mass parameter  $M$  of black hole. Now if one considers that the change of mass of a black hole is due to the Hawking radiation [12] only, then here  $M$  can be taken as independent of  $v$  as extremal black hole has temperature zero and hence does not radiate. Then we have  $V = \pi M^2 v$ ; i.e. the maximum interior volume grows linearly with advanced time.

Let us now discuss the difference between the present calculation and the earlier one [14]. Earlier in [14], it has been shown that the quantity  $dV/dv$  vanishes for ERN black hole. In that calculation, the author first calculated the quantity for non-ERN metric, and then the extremal limit  $Q \rightarrow M$  was taken. Here we have taken the extremal black hole as a starting point. As a result, the difference between these two approaches is lying in the factor inside the square root of Eq. (6). For non-extremal case it was  $(2M/r - Q^2/r^2 - 1)$ , which for extremal limit reduces to  $[-(M-r)^2/r^2]$ . Then the square root of it gives us an imaginary quantity. Whereas the crucial point in our way of handling the problem is to consider the ERN metric from the beginning. As a result the metric coefficient reduces to  $(M-r)^2/r^2$  which is positive on both sides of the horizon  $r = M$ . Now the square root of it gives us two values depending on the sign. But since we are interested in region  $r < M$ , one has to take a particular sign to get the volume a positive quantity. This is precisely has been done here. As a result of it we are getting a non-zero value of the relevant quantity. Interestingly, it is exactly equal to the horizon entropy.

---

<sup>1</sup>The reason for choosing  $r = \text{constant}$  surface has been discussed in [9] extensively. It was shown by extremizing the Lagrangian corresponding to the volume as well as by numerical analysis.

## 2.2 Extremal Kerr black hole

The metric of the extremal Kerr metric is given by

$$ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\rho^2} dv^2 + 2dvdr + \rho^2 d\theta^2 + \frac{A \sin^2 \theta}{\rho^2} d\phi^2 - 2a \sin^2 \theta dr d\phi - \frac{4a^2 r}{\rho^2} \sin^2 \theta dv d\phi, \quad (9)$$

where  $\Delta = (r-a)^2$ ,  $\rho^2 = r^2 + a^2 \cos^2 \theta$  and  $A = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta$ . This has been obtained by imposing the extremal limit  $M = a$  in the original metric [20]. Here  $a$  is the angular momentum of the black hole per unit mass and the event horizon is at  $r = a$ . Following the earlier argument, one can show that the interior volume is

$$V = \int dv d\theta d\phi (a - r)(r^2 + a^2 \cos^2 \theta)^{1/2} \sin \theta. \quad (10)$$

Performing the integrations on angular coordinates we obtain

$$V = \pi \int^v dv a^2 \left(1 - \frac{r}{a}\right) \left[ 2\sqrt{1 + \frac{r^2}{a^2}} + \frac{r^2}{a^2} \ln \left( \frac{\sqrt{1 + \frac{r^2}{a^2}} + 1}{\sqrt{1 + \frac{r^2}{a^2}} - 1} \right) \right]. \quad (11)$$

Now we need to find the value of  $r$  at which the integrand is maximum. Since, the radial coordinate  $r$  runs from zero to infinity, a plot between the integrand as a function of  $r/a$  shows that it has maxima at  $r/a = 0$  (See the Figure 1 below. Here  $G(r/a) = \left(1 - \frac{r}{a}\right) \left[ 2\sqrt{1 + \frac{r^2}{a^2}} + \frac{r^2}{a^2} \ln \left( \frac{\sqrt{1 + \frac{r^2}{a^2}} + 1}{\sqrt{1 + \frac{r^2}{a^2}} - 1} \right) \right]$ ).

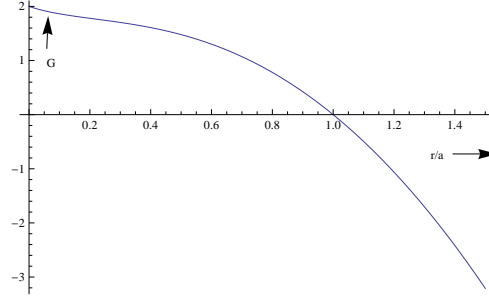


Figure 1:  $G(r/a)$  vs  $r/a$  plot.

Substituting this, we find the maximum interior volume as

$$V = 2\pi \int^v a^2 dv. \quad (12)$$

Hence  $dV/dv = 2\pi a^2$ , which is again one quarter of the horizon area. Whereas, in an earlier calculation [10], this has been shown to be zero in the extremal limit.

What have we achieved in this section? It may be mentioned that the result might play an important role to explore the origin of entropy of extremal black holes. The existing analysis [17, 18, 19] suggests that these contain entropy. Also, recently [11] for Schwarzschild case, the entropy has been calculated from the phase-space approach corresponding to its maximum interior volume. All these suggests that the non-zero value of entropy for an extremal black hole is not compatible to the vanishing of this volume. Here we showed that a careful calculation leads to non-zero value and so it can contain enough information which leads to the horizon entropy of an extremal black hole.

### 3 $(1 + D)$ dimensional Schwarzschild black hole

Here the dependence of the CR volume on the spacetime dimensions will be extensively discussed for an arbitrary dimensional Schwarzschild black hole. Considering the Hawking evaporation, we shall observe that at large dimensions the volume almost remains constant with respect to the advanced time.

The  $(1 + D)$  dimensional Schwarzschild black hole metric is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{(D-1)}^2, \quad (13)$$

where  $f(r) = 1 - (r_H/r)^{D-2}$  with the horizon radius is  $r_H = \left[ \frac{16\pi M}{(D-1)A_{D-1}} \right]^{\frac{1}{D-2}}$ . Here  $M$  is the mass of the black hole and  $A_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}$ . The horizon area is found to be

$$\mathcal{A} = A_{D-1} r_H^{D-1}. \quad (14)$$

To use the CR formulation [9], first of all we need to transform the metric (13) in Eddington-Finkelstein co-ordinates  $(v, r, \theta, \phi)$ :

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega_{(D-1)}^2, \quad (15)$$

where  $v = t + r^*$  with  $dr^* = dr/f(r)$ . Now the maximum volume inside the horizon is defined as the value of the maximum of the volume of a  $D$ -dimensional surface  $\Sigma$ , which is a direct product of a  $(D-1)$ -sphere and a curve on the  $(v-r)$ -plane; i.e.  $\Sigma = \gamma \times S^{(D-1)}$ . According to [9], the maximum volume corresponds to  $r = \text{constant}$  surface. Then from (15), the proper volume of this surface is given by

$$\begin{aligned} V &= \int_V \sqrt{-g} \, d\Omega' dv \\ &= \int_V \sqrt{r^{2(D-1)} f(r)} \, d\Omega' dv \end{aligned} \quad (16)$$

Now inside the horizon the metric coefficient is given by  $f(r) = (r_H/r)^{D-2} - 1$  and so the integration over the angular coordinates can be done. Therefore the volume turns out to be

$$V = A_{D-1} \int^v r^{D-1} \sqrt{\left(\frac{r_H}{r}\right)^{D-2} - 1} \, dv. \quad (17)$$

Then the maximum volume inside the horizon will be given by the value of above integration for a particular  $r$  at which the integrand is maximum. It is found that this value of  $r$  is  $r_c = r_H \left(\frac{D}{2(D-1)}\right)^{\frac{1}{D-2}}$ . Substituting this in (18) we obtain the maximum volume inside the horizon as

$$V = A_{D-1} \left(\frac{D}{2(D-1)}\right)^{\frac{D-1}{D-2}} \left(\frac{D-2}{D}\right)^{1/2} \int^v r_H^{D-1} dv. \quad (18)$$

Note that in the above the  $r_H$  term can not be kept outside the integration as  $r_H$  can change with the advanced time  $v$ . In the below we shall consider the Hawking evaporation of the black hole, in which case the mass is changing with  $v$ . Substituting the value of  $r_H$  in terms of mass  $M$ , one obtains the variation of the CR volume with respect to the Eddington-Finkelstein time as

$$\frac{dV}{dv} = L(D) M^{\frac{D-1}{D-2}}, \quad (19)$$

where  $L(D) = (\frac{D}{2(D-1)})^{\frac{D-1}{D-2}} (\frac{D-2}{D})^{1/2} (\frac{16\pi}{D-1})^{\frac{D-1}{D-2}} A_{D-1}^{\frac{1}{2-D}}$ . Note that for  $D \geq 3$  the right hand side of the above is always positive. This implies that the interior volume is always increasing, even if the mass of the black hole is decreasing. The same was also concluded in [14] for  $(1+3)$  dimensional Schwarzschild black hole. Therefore we can say that this feature is independent of the dimensions of spacetime. The new observation, for the present case, is the following. Here the value of right hand side depends not only the value of mass, but also on the space dimensions  $D$ . Now if one plots this as a function of  $D$  with a fixed value of mass, then the nature of  $dV/dv$  is the following (See Figure 2).

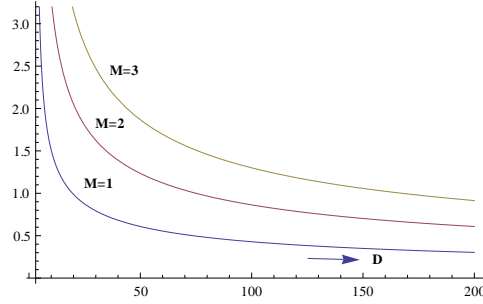


Figure 2:  $\frac{dV}{dv}$  vs  $D$  plot.

This shows that the rate of increase of volume decreases with the increase of spacetime dimensions; i.e. the volume increase decelerates. It implies that at sufficiently large  $D$ , the increment of interior volume is almost negligible. This fact will also be again discussed in the next part of paper by calculating the total volume change for the time interval of total evaporation by Hawking process.

Now if we consider only Hawking radiation as the cause of change in black hole mass, then the flux is given by [21]

$$\frac{dM}{dv} = -\frac{B(D)}{r_H^2}, \quad (20)$$

where  $B(D) = \frac{(D+1)(D-2)}{2} (\frac{D-2}{4\pi})^{D+1} (\frac{D}{2})^{\frac{D-1}{D-2}} (\frac{D}{D-2})^{\frac{D-1}{2}} \frac{D\zeta(D+1)\hbar}{\pi}$ . The expression has been obtained by Eikonal approximation and is valid for large frequency; i.e. in the limit  $D \gg 1$ . If the initial black hole mass is  $M_0$ , then after time  $v$  the expression for black hole mass can be obtained by integrating (20). It turns out to be

$$M(v) = \left( -P(D)v + M_0^{\frac{D}{D-2}} \right)^{\frac{D-2}{D}}, \quad (21)$$

with  $P(D) = B(D) (\frac{(D-1)A_{D-1}}{16\pi})^{\frac{2}{D-2}} \frac{D}{D-2}$ . Next, replacing the above value of black hole mass in (19), we obtain the rate of volume change as a function of advance time  $v$ :

$$\frac{dV}{dv} = L(D) \left( -P(D)v + M_0^{\frac{D}{D-2}} \right)^{\frac{D-1}{D}} \quad (22)$$

Also note that if the complete evaporation time is  $v_{max}$  by Hawking process, then its value is can be found out by setting  $M = 0$  in (21). It is given by

$$v_{max} = \frac{M_0^{\frac{D}{D-2}}}{P(D)}. \quad (23)$$

Within this time, the total volume change, calculate from (22), is

$$\begin{aligned} \int_{V_i}^{V_f} dV &= L(D) \int_0^{v_{max}} (-P(D)v + M_0^{\frac{D}{D-2}})^{\frac{D-1}{D}} dv \\ \Rightarrow \Delta V &= V_f - V_i = Q(D) M_0^{\frac{2D-1}{D-2}}, \end{aligned} \quad (24)$$

where the value of  $Q(D)$  is given by

$$\begin{aligned} Q(D) &= \frac{L(D)}{P(D)} \frac{D}{2D-1} \\ &= (16\pi)^{\frac{D+1}{D-2}} \left(\frac{1}{D-1}\right)^{\frac{2D}{D-2}} \left(\frac{D-2}{D}\right)^{D/2} \left(\frac{4\pi}{D-2}\right)^{D+1} \\ &\quad \times \frac{2\pi}{D(D+1)(2D-1)\zeta(D+1)\hbar} (A_{D-1})^{\frac{3}{2-D}}. \end{aligned} \quad (25)$$

Let us now plot  $\Delta V$  as a function of  $D$  to reveal the nature of volume at the different dimensions.

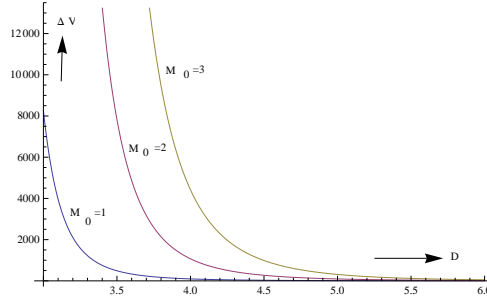


Figure 3:  $\Delta V$  vs  $D$  plot. Here  $\Delta V$  is taken in units of  $\hbar$ .

Figure 3 shows that the total volume change decreases with the increase of space dimensions. For very large value of  $D$ , the quantity is almost negligible. This implies that for a sufficiently higher dimensional Schwarzschild black hole the maximum interior volume remains almost same even if it is radiating through Hawking process.

The possible reason behind the decelerated increase of volume can be explained as follows. If we look at the (1+3) dimensional case with Hawking radiation taking into account, the flux (rate of energy emission from the black hole) is increasing with the decrease of mass ( $dM/dv \sim 1/M^2$ ) and correspondingly the rate of volume increment decreases as  $dV/dv \sim M^2$ ; i.e. at late time the flux increases whereas the volume increases with decreased rate. The similar is also happening with the analysis in the higher dimensions. At higher dimensions the flux increases (see Figure 4 corresponding to Eq. (20))

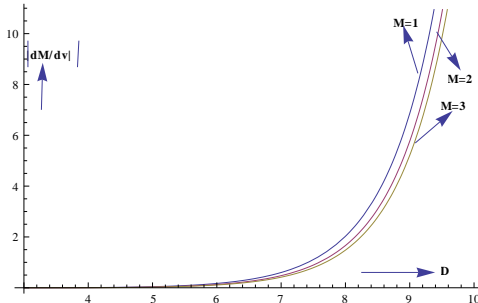


Figure 4:  $|dM/dv|$  vs  $D$  plot. Here  $|dM/dv|$  is taken in units of  $\hbar$ .

and so correspondingly we obtained decelerated increase of interior volume. It implies that the increase of Hawking flux reflects a negative impact on the maximum interior volume.

## 4 Discussions

The motivations of this paper were two fold: (a) First to calculate the CR volume for the extremal black holes; (b) Finally, the investigation of the nature of CR volume with the dimensions of spacetime. Here we have discussed both the options for certain black hole spacetimes. The results, obtained here, are as follows.

Our analysis showed that, if we our starting metric is strictly extremal, then the CR volume is not vanishing. Rather it has been observed that the change in this quantity with respect to the advanced time is one quarter to the horizon area, which precisely is the entropy of the extremal hole. This we have shown explicitly for ERN and EK cases. On a closer look we also found that this rate is always positive. So at any point of advanced time we are supposed to get non zero volumes for both these cases. This result clearly contradicts with the existing results [14, 10] which predict the zero volume in the extremal limit. The interesting fact of the non-zero value is that this is more compatible with the suggested non zero entropy of extremal black holes. The inside volume of horizon has enough space to contain sufficient information which leads to the require entropy of the horizon.

Next, considering only Hawking radiation as the reason of change in black hole mass, we observed that for a  $(1 + D)$  dimensional Schwarzschild black hole the rate of increase in CR volume is always positive, suggesting a nonzero interior volume even at the end of Hawking radiation process. This is similar to the result found for  $(1 + 3)$  dimensional case [9, 14]. It is interesting to note that at the end of Hawking radiation process when mass of the black hole touches the zero value, indicating the horizon surface area is also; but the CR volume remains non zero, leaving a possibility of solving information loss paradox with black hole remnants. In this paper, we got an additional feature in the change of black hole interior volume. It has been observed that as we go to higher dimensions keeping the black hole mass constant, the volume increase gets decelerated. Here we have also estimated the total change in CR volume  $(V_f - V_i)$  during the Hawking radiation process in terms of its initial mass and we have again seen that this estimated volume change also decreases for the increase in dimensions. So from these results we can assert that if we go to very higher dimensions we are supposed to get an almost negligible volume increase rate or constant CR volume of a Schwarzschild black hole. The precise physical significance of which is unknown to us.

Now we want to bring some important facts for higher dimensional case which may have connection with our analysis. From recent analysis [22] it has been observed that the time difference between the emission of two consecutive quanta in Hawking radiation process decreases with the increase of number of dimensions. The typical value of  $D$  for continuous spectrum was found out to be  $D \geq 10$ . Then one may be curious to know if there is any physical connection between the continuous emission with the decelerated rate of volume increase with increasing dimensions. Another observation can also be important in this direction. The perturbative analysis for non-uniform black strings suggests that for  $D > 13$  they are stable and so these can be the end point of Gregory-Laflamme instability (see the review [23]). Therefore it might be possible that this nonuniform nature can be expected to offer a resistance in the rate of volume increase of the black hole.

Finally, we must mention that the above discussion is completely speculative. To reach at the concrete conclusion, more analysis is required. Moreover, the obtained result – the maximum interior volume for an extremal black hole is exactly equal to its horizon entropy – is just a suggestion. In order to make any conclusive statement further investigation is needed for other extremal black holes in other theories of gravity.



## Acknowledgments

The research of one of the authors (BRM) is supported by a START-UP RESEARCH GRANT (No. SG/PHY/P/BRM/01) from Indian Institute of Technology Guwahati, India.

## References

- [1] M. K. Parikh, “The Volume of black holes,” *Phys. Rev. D* **73**, 124021 (2006) [hep-th/0508108].
- [2] D. Grumiller, “The Volume of 2-D black holes,” *J. Phys. Conf. Ser.* **33**, 361 (2006) [gr-qc/0509077].
- [3] W. Ballik and K. Lake, “The volume of stationary black holes and the meaning of the surface gravity,” arXiv:1005.1116 [gr-qc].
- [4] W. Ballik and K. Lake, “Vector volume and black holes,” *Phys. Rev. D* **88**, no. 10, 104038 (2013) [arXiv:1310.1935 [gr-qc]].
- [5] B. S. DiNunno and R. A. Matzner, “The Volume Inside a Black Hole,” *Gen. Rel. Grav.* **42**, 63 (2010) [arXiv:0801.1734 [gr-qc]].
- [6] T. K. Finch, “Coordinate families for the Schwarzschild geometry based on radial timelike geodesics,” *Gen. Rel. Grav.* **47**, no. 5, 56 (2015) [arXiv:1211.4337 [gr-qc]].
- [7] M. Cvetič, G. W. Gibbons, D. Kubiznak and C. N. Pope, “Black Hole Enthalpy and an Entropy Inequality for the Thermodynamic Volume,” *Phys. Rev. D* **84**, 024037 (2011) [arXiv:1012.2888 [hep-th]].
- [8] G. W. Gibbons, “What is the Shape of a Black Hole?,” *AIP Conf. Proc.* **1460**, 90 (2012) [arXiv:1201.2340 [gr-qc]].
- [9] M. Christodoulou and C. Rovelli, “How big is a black hole?,” *Phys. Rev. D* **91**, no. 6, 064046 (2015) [arXiv:1411.2854 [gr-qc]].
- [10] I. Bengtsson and E. Jakobsson, “Black holes: Their large interiors,” *Mod. Phys. Lett. A* **30**, no. 21, 1550103 (2015) [arXiv:1502.01907 [gr-qc]].
- [11] B. Zhang, “Entropy in the interior of a black hole and thermodynamics,” *Phys. Rev. D* **92**, no. 8, 081501 (2015) [arXiv:1510.02182 [gr-qc]].
- [12] S. W. Hawking, “Particle Creation by Black Holes,” *Commun. Math. Phys.* **43**, 199 (1975) Erratum: [*Commun. Math. Phys.* **46**, 206 (1976)].
- [13] Y. C. Ong, “Never Judge a Black Hole by Its Area,” *JCAP* **1504**, no. 04, 003 (2015) [arXiv:1503.01092 [gr-qc]].
- [14] Y. C. Ong, “The Persistence of the Large Volumes in Black Holes,” *Gen. Rel. Grav.* **47**, no. 8, 88 (2015) [arXiv:1503.08245 [gr-qc]].
- [15] Y. C. Ong, “Black Hole: The Interior Spacetime,” arXiv:1602.04395 [gr-qc].
- [16] M. Christodoulou and T. De Lorenzo, “On the volume inside old black holes,” arXiv:1604.07222 [gr-qc].

- [17] A. Sen, “Black hole entropy function and the attractor mechanism in higher derivative gravity,” JHEP **0509**, 038 (2005) [hep-th/0506177].
- [18] A. Sen, “Black Hole Entropy Function, Attractors and Precision Counting of Microstates,” Gen. Rel. Grav. **40** (2008) 2249 [arXiv:0708.1270 [hep-th]].
- [19] B. R. Majhi, “Entropy function from the gravitational surface action for an extremal near horizon black hole,” Eur. Phys. J. C **75**, no. 11, 521 (2015) [arXiv:1503.08973 [gr-qc]].
- [20] R. P. Kerr, “Gravitational field of a spinning mass as an example of algebraically special metrics,” Phys. Rev. Lett. **11**, 237 (1963).
- [21] S. Hod, “Bulk emission by higher-dimensional black holes: Almost perfect blackbody radiation,” Class. Quant. Grav. **28**, 105016 (2011) [arXiv:1107.0797 [gr-qc]].
- [22] S. Hod, “The Hawking cascades of gravitons from higher-dimensional Schwarzschild black holes,” Phys. Lett. B **756**, 133 (2016) [arXiv:1605.08440 [gr-qc]].
- [23] H. S. Reall, “Higher dimensional black holes,” Int. J. Mod. Phys. D **21**, 1230001 (2012) [arXiv:1210.1402 [gr-qc]].